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LETTER TO THE EDITOR

Cyclic rotations, contractibility and Gauss–Bonnet

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Abstract. Let a rigid object or frame of reference have identical initial and final orientations but be rotated in any way in between, with an angular velocity $\boldsymbol{\omega}(t)$. Any unit vector $\boldsymbol{u}(t)$ carried with the frame passes through a cycle of directions enclosing a solid angle Ω. The full relation between these three quantities is shown to be $2\pi n = \Omega + \int \boldsymbol{\omega}\cdot\boldsymbol{u}\,\mathrm{d}t$, mod 4π, where the turn number n is zero if the sequence of orientations of the frame is contractible and unity if it is non-contractible. The main derivation uses the Calugareanu relation, $Lk = Wr + Tw$, between linking number, writhe, and twist of a ribbon loop. An outline alternative derivation uses the Berry phase of a quantum spin $\frac{1}{2}$. Finally the result is applied to the standard parallel transport holonomy expressed in the Gauss–Bonnet theorem: it is refined to be correct mod 4π rather than merely mod 2π.

Let a rigid object or frame of reference have identical initial and final orientations but be rotated in any way in between, with an angular velocity $\boldsymbol{\omega}(t)$. Any unit vector $\boldsymbol{u}(t)$ carried with the frame passes through a cycle of directions enclosing a solid angle Ω. The full relation between these three quantities will be shown to be

$$2\pi n = \Omega + \int \boldsymbol{\omega}\cdot\boldsymbol{u}\,\mathrm{d}t \bmod 4\pi \tag{1}$$

where the turn number n is zero if the sequence of orientations of the frame is contractible and unity if it is non-contractible. The result (1) reduced to mod 2π, that is, with the term $2\pi n$ replaced by zero, is essentially well known [1–3], for example, being associated with the standard parallel transport holonomy expressed in the Gauss–Bonnet theorem (see final paragraph below). However, the solid angle is defined mod 4π, not mod 2π, and this invites refinement of the equation to mod 4π, with (1) as the outcome.

The main derivation of (1), given now, is geometric, using the Calugareanu relation (2) for a ribbon loop in space [4–6]. (An alternative derivation using the Berry phase of quantum spin $\frac{1}{2}$ is outlined near the end.)

$$Lk = Wr + Tw. \tag{2}$$

The origin of this relation can be indicated once the meanings of the terms are explained in the next few paragraphs. The integer Lk is the number of 2π twists in the ribbon, counting the right-hand screw sense as positive (beware, however, the different technical meaning of 'twist' below). The ribbon, which is supposed indefinitely narrow, has two edges which are nearly parallel loops in space and Lk is the number of times one loop winds around the other; that is, their 'linking number' defined by an integral around each loop

$$Lk = \frac{1}{4\pi}\oint\oint \frac{(\boldsymbol{r}-\boldsymbol{r}')\cdot(\mathrm{d}\boldsymbol{r}_\wedge \mathrm{d}\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}. \tag{3}$$

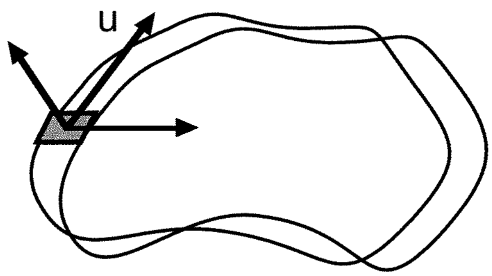

Figure 1. Ribbon loop with the natural orthogonal frame defined by the unit tangent vector $\boldsymbol{u}(t)$ at position t along it, the ribbon unit normal vector and and the third orthogonal unit vector. The ribbon shown, being untwisted, has $Lk = 0$ but $Tw \neq 0$, probably.

This can be understood as the line integral around one loop of the magnetic field due to a hypothetical current μ_0^{-1} in the other.

For the 'twist' Tw the natural othogonal frame is defined at any point on the ribbon (figure 1) based on the unit tangent vector $\boldsymbol{u}$, the ribbon normal and their cross product. If t is a coordinate along the ribbon (which will later be specialized to arc length), increasing in the direction of $\boldsymbol{u}$, then

$$Tw = \frac{1}{2\pi} \int \boldsymbol{\omega} \cdot \boldsymbol{u} \, \mathrm{d}t \tag{4}$$

where $\boldsymbol{\omega}$ is the 'angular velocity' vector of the frame with respect to t, and the integral is over the whole length of the ribbon. The quantity Tw is a real number which depends only on the shape of the ribbon, not the coordinatisation.

'Writhe' Wr is only a property of the central axis loop of the ribbon which lies midway between the edge loops. Once this axis loop is defined the ribbon itself can be discarded as far as writhe is concerned. It is defined algebraically as self-linking: the integral (3) with the two loops coincident along the ribbon central axis

$$Wr = \frac{1}{4\pi} \oint \oint \frac{(\boldsymbol{r} - \boldsymbol{r}') \cdot (\mathrm{d}\boldsymbol{r}_\wedge \mathrm{d}\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}. \tag{5}$$

This is again a real number with a geometric interpretation. The chords of the axis loop form a two-dimensional family parametrized by t and t', the endpoint coordinates. The chord directions, or unit vectors, thus form a net on the unit sphere (figure 2) and Wr is the total area of the net divided by 4π. The net is bounded by two closed curves which are mutual inverses through the sphere centre. These curves represent the tangent vectors $\boldsymbol{u}(t)$, and its reverse, which are the limiting chord unit vectors as $t \to t'$ for $t < t'$ and $t > t'$. Being mutual inverses, the solid angles enclosed by the two bounding curves are $\pm\Omega \bmod 4\pi$. The difference of their solid angles is the solid angle in between them, $4\pi Wr \bmod 4\pi$. Therefore $Wr = \Omega/2\pi \bmod 1$.

This fact was initially observed by Fuller [7], using a different argument, and indeed refined to the form required here

$$Wr = 1 + \Omega/2\pi \bmod 2. \tag{6}$$

This refined form follows from recognizing that (6) is true for a circle ($Wr = 0, \Omega = 2\pi$), and Wr changes continuously under deformation except for jumps of 2 when the loop crosses itself. That concludes the explanation of the meanings of the terms Lk, Wr, and Tw in (2). To derive (2) one takes, explicitly, the intended limit of an infinitely narrow ribbon in the integrand of (3) for Lk. Convert (3) to an integral over the ribbon axis coordinates t and

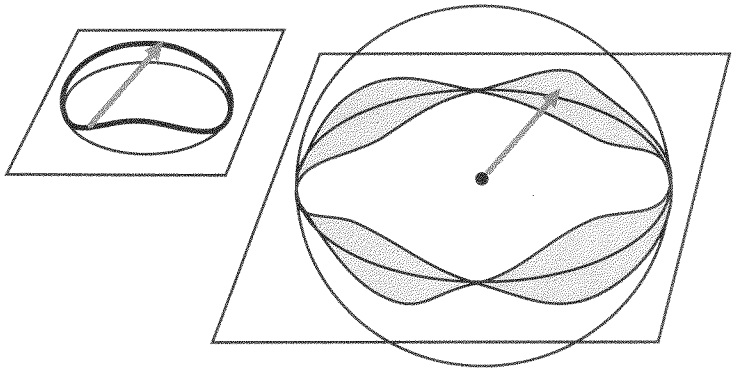

Figure 2. The ribbon is no longer shown, but its central axis loop is pictured on the left in bold as a slightly bent circle. The directions of all possible chords of the loop are represented by unit vectors in the unit sphere on the right, with tips lying in the shaded area. The two chord endpoint positions on the loop, (t, t'), thus form a coordinate net (not shown) filling this shaded area. The two boundary lines of the net are formed by the set of loop unit tangent vectors and their reverses. For a more bent loop the net could have folds which could cross the boundary lines and indeed render the whole sphere shaded.

t' by associating, with each value of coordinate, the vectors $\boldsymbol{r}'$ and $\boldsymbol{r}$ of the closest point on each ribbon edge. Thus $\mathrm{d}\boldsymbol{r} = \dot{r}(t)\mathrm{d}t$, and $\mathrm{d}\boldsymbol{r}' = \dot{r}(t')\mathrm{d}t'$. The term $1/|\boldsymbol{r}'(t') - \boldsymbol{r}(t)|^3$ becomes, in the limit, $1/|\boldsymbol{r}(t') - \boldsymbol{r}(t)|^3 + (2/|\boldsymbol{r}'(t) - \boldsymbol{r}(t)|^2|\dot{r}(t)|)\delta(t' - t)$. In the integral, the second term here can be interpreted as Tw directly, and setting $\boldsymbol{r}'(t') = \boldsymbol{r}(t')$ in the numerator in (3), the first gives Wr (expressed in terms of t and t').

The proof of (1) follows by considering a frame turning but also translating and thereby generating a ribbon. The frame is to be described by the unit vector $\boldsymbol{u}$ and any orthogonal vector **v** whose magnitude will be taken indefinitely small. As the frame turns, these vectors rotate, but also their common origin translates with unit velocity $\boldsymbol{u}(t)$, so that the time t measures arc length along its path. The ribbon is to be the surface in space swept out by the vector $\boldsymbol{v}(t)$. It can then be extended to form a closed loop by adding a flat 'return' section of ribbon joining the final to the initial vectors $\boldsymbol{v}$ and lying in their plane. Its shape is unimportant except that it should not cross itself, and the tangent vectors should be parallel at the two ends, indeed both perpendicular to the vectors $\boldsymbol{v}$ (and in their plane).

The result (1) emerges on writing down the Calugareanu relation (2) for this ribbon loop. The twist Tw of the ribbon is $\int \boldsymbol{\omega} \cdot \boldsymbol{u}\, \mathrm{d}t/2\pi$ for the outward section where $\boldsymbol{\omega}$ is the angular velocity vector of the frame, plus zero for the return section. Likewise, the solid angle enclosed by the ribbon axis tangent vector $\boldsymbol{u}$ is Ω for the outward section, plus zero for the return section, plus a total of 2π from the sharp turns at each end. Thus using (6) the quantity ($Wr \bmod 2$) of the ribbon equals $1 + (\Omega + 2\pi)/2\pi = \Omega/2\pi \bmod 2$.

There remains the linking number Lk of the ribbon edge loops. Only the value of $Lk \bmod 2$ is required, and this value is shared by any deformation of the ribbon loop because, as Fuller notes, the value of Lk jumps by 2 if the ribbon passes through itself. Thus, to find $Lk \bmod 2$ the outward section of the ribbon can be 'pulled' straight. Its value is then the number of twists mod 2 in the outward section. However, recalling Dirac's 'belt trick' [8, 9], this is zero if the sequence of orientations passed through by the unit sphere was contractible, and unity if it is non-contractible in the space of orientations SO(3). That is, $Lk \bmod 2 = n$, the turn number of the frame. Combining the three terms Lk, Tw, and Wr, all mod 2, to form the Calugareanu relation (mod 2) yields (1) as claimed.

The form of the result (1) invites an alternative derivation via the quantum mechanics of spin $\frac{1}{2}$ which runs, in outline, as follows. The unitary operator corresponding to the cyclic rotation of a spin $\frac{1}{2}$ is known to be the identity operator times $(-1)^n$, where n has the same meaning as previously. This rotation is achieved by a time-dependent Hamiltonian $H = \frac{1}{2}\hbar\boldsymbol{\omega}(t)\cdot\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is the vector of Pauli matrices. The Hamiltonian drives a general spin $\frac{1}{2}$ state along a path described, up to overall phase, by a density matrix $\rho = \frac{1}{2}(1 + \boldsymbol{u}(t)\cdot\boldsymbol{\sigma})$. The total phase change of a state can be split up in the manner of Aharonov and Anandan [10] into the geometric, Berry phase [11] plus a remaining 'dynamical' phase. The total phase is $n\pi$, the Berry phase is minus half the solid angle Ω enclosed by $\boldsymbol{u}(t)$, and the dynamical phase is $-\int \mathrm{Tr}(H\rho/\hbar)\,\mathrm{d}t = -\frac{1}{2}\int \boldsymbol{\omega}\cdot\boldsymbol{u}\,\mathrm{d}t$. Thus, $\mathrm{mod}\,2\pi$, one has $n\pi = \frac{1}{2}\Omega + \frac{1}{2}\int \boldsymbol{\omega}\cdot\boldsymbol{u}\,\mathrm{d}t$, which is (1).

Finally it can be noted how the formula (1) applies to parallel transport holonomy. Consider a smooth closed curve drawn on a smooth curved surface, with arc length along it denoted by t. An orthogonal frame at each point along the circuit is defined by surface normal unit vector $\boldsymbol{u}$ and curve tangent vector $\boldsymbol{v}$. The frame has an 'angular velocity' vector with respect to change in the position t, and the integral $\int \boldsymbol{\omega}\cdot\boldsymbol{u}\,\mathrm{d}t$, equals the angle $\theta(t)$ that the frame has rotated (about the vector $\boldsymbol{u}$) with respect to a frame parallel transported along the curve. After a complete circuit, as is well known from the Gauss–Bonnet theorem θ is equal to (minus) the integrated Gaussian curvature of the enclosed area of surface, which, in turn is (minus) the solid angle Ω enclosed by the motion of the vector $\boldsymbol{u}$: $\theta + \Omega = 0\,\mathrm{mod}\,2\pi$. However, from (1) this result can be refined to

$$2\pi n = \Omega + \theta\,\mathrm{mod}\,4\pi \tag{7}$$

where the turn number n is 0 if the sequence of frame orientations is contractible and 1 otherwise.

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